Directional Derivative in direction u

of a fcn $g(x,y)$

$$= \frac{\partial g}{\partial u} = u(g) = D_u(g) = \nabla g \cdot u$$

= rate of change of g as you move from (x,y) in the u direction (velocity u).

= 0 if u is perp to ∇g

ie. u is $\parallel$ to level sets.

= **max** when u is $c\nabla g$ for some constant $c > 0$

= **min** when u is $-c\nabla g$ for some $c > 0$.

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(2) Find the critical points of

$$h(x,y) = \left(\tfrac{1}{2} - x^2 + y^2\right) e^{1-x^2-y^2}$$

$\nabla h = (0,0)$ or (undefined)

$= (h_x, h_y)$

$h_y = 2y e^{\sim} + \left(\tfrac{1}{2} - x^2 + y^2\right) e^{\sim}(-2y)$

$2y e^{\sim}\left(1 - \tfrac{1}{2} + x^2 - y^2\right)$

$$= \left(-2x e^{1-x^2-y^2}\left(1 + \left(\tfrac{1}{2} - x^2 + y^2\right)\right),\ 2y e^{1-x^2-y^2}\left(\tfrac{1}{2} + x^2 - y^2\right)\right)$$

$$= \left(-2x\left(\tfrac{3}{2} - x^2 + y^2\right)e^{1-x^2-y^2},\ 2y\left(\tfrac{1}{2} + x^2 - y^2\right)e^{1-x^2-y^2}\right)$$

$$= (0, 0)$$

$$\begin{cases} -2x\left(\tfrac{3}{2} - x^2 + y^2\right)\underbrace{e^{1-x^2-y^2}} = 0 \\ 2y\left(\tfrac{1}{2} + x^2 - y^2\right)\underbrace{e^{1-x^2-y^2}}_{\text{always} > 0} = 0 \end{cases}$$

divide by them

☆ $-2x\left(\tfrac{3}{2} - x^2 + y^2\right) = 0$

☆☆ $2y\left(\tfrac{1}{2} + x^2 - y^2\right) = 0$

☆ $-2x = 0$ OR $\tfrac{3}{2} - x^2 + y^2 = 0$

$\Rightarrow x = 0$ ☆☆ $x^2 - y^2 = \tfrac{3}{2}$ hyperbola

$\Rightarrow$ ☆☆ $2y\left(\tfrac{1}{2} + 0 - y^2\right) = 0$

$\Rightarrow y=0$ $\boxed{\text{OR}}$ $\frac{1}{2}-y^2=0$

$\Downarrow$ $\quad y^2=\frac{1}{2}$

$(0,0)$ $\quad y=\pm\frac{1}{\sqrt{2}}$

$\left(0,\frac{1}{\sqrt{2}}\right)$

$\left(0,-\frac{1}{\sqrt{2}}\right)$

$2y\left(\frac{1}{2}+x^2-y^2\right)=0$

$\Rightarrow 2y\left(\frac{1}{2}+\frac{3}{2}\right)=0$

$4y=0$

$y=0$

$x^2=\frac{3}{2}$

$x=\pm\sqrt{\frac{3}{2}}$

$\left(\sqrt{\frac{3}{2}},0\right), \left(-\sqrt{\frac{3}{2}},0\right)$

the critical pts of

$$h(x,y)=\left(\frac{1}{2}-x^2+y^2\right)e^{1-x^2-y^2}$$

are $(0,0), \left(0,\frac{1}{\sqrt{2}}\right), \left(0,-\frac{1}{\sqrt{2}}\right), \left(\sqrt{\frac{3}{2}},0\right), \left(-\sqrt{\frac{3}{2}},0\right)$

Note – if we graph $z=h(x,y)$, we should see these 5 pts where the tangent plane is horizontal (i.e. parallel to the xy-plane)

Example: Find the critical points of

$$A(x,y,z)=9-\left(x(y-1)(z+2)\right)^2.$$

$\boxed{\underline{\text{Solution:}}}$ $$\nabla A = \Big(-2\big(x(y-1)(z+2)\big)^1 (y-1)(z+2),$$
$$-2\big(x(y-1)(z+2)\big)^1 x(z+2),$$
$$-2\big(x(y-1)(z+2)\big)^1 x(y-1)\Big)$$

$$\Rightarrow \nabla A = \left(-2x(y-1)^2(z+2)^2, -2x^2(y-1)(z+2)^2, -2x^2(y-1)^2(z+2)\right)$$
$$= (0,0,0)$$

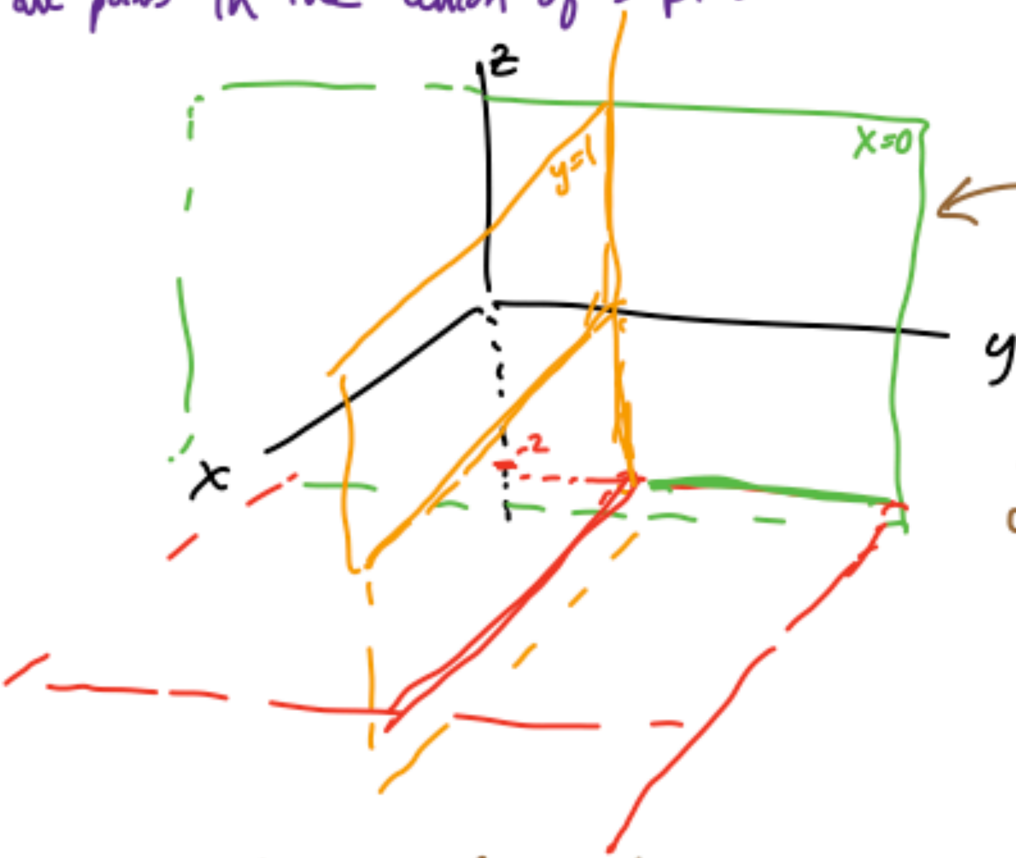
$$-2x(y-1)^2(z+2)^2 = 0 \quad \star$$
$$-2x^2(y-1)(z+2)^2 = 0 \quad ✓✓$$
$$-2x^2(y-1)^2(z+2) = 0 \quad G$$

$\star$ $x=0$ $\boxed{\text{OR}}$ $y-1=0$ $\boxed{\text{OR}}$ $z+2=0$

✓✓ G ✓✓ G ✓✓ G

So $\left\{ \begin{matrix} x=0 \\ y, z \text{ can be anything} \end{matrix} \right\}$ plane or $\left\{ \begin{matrix} y=1 \\ x, z \text{ can be anything} \end{matrix} \right\}$ plane or $\left\{ \begin{matrix} z=-2 \\ x, y \text{ can be anything} \end{matrix} \right\}$ plane.

In this case the critical points are are points in the union of 3 planes.



every point of these 3 planes are critical points.

This actually makes sense.

$$A(x,y,z) = 9 - \left(x(y-1)(z+2)\right)^2.$$

This is maximum when $A = 9$

i.e. $x(y-1)(z+2) = 0$

i.e. $x=0$ or $y=1$ or $z=-2$.

So all of our critical points are maxima.

The function's value goes down as you

move away from those 3 planes!

<u>New Question</u>: If we find a critical point of a function, is there a way to determine what type of critical point it is?
(local max, local min, saddle pt.)
relative max   relative min.

Solution: use 2^{nd} derivatives

Aside: Some Linear Algebra (Matrices)

Calculating eigenvalues of a matrix.

$n \times n$ matrix $\longrightarrow$ we will calculate n eigenvalues. When the matrix is symmetric, they will all be real numbers.

Technique for finding eigenvalues:

$n \times n$ A

$$M = (A - \underset{\substack{\uparrow \\ \text{constant - we will solve for it.}}}{\lambda} I) = A - \begin{pmatrix} \lambda & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \cdots \\ 0 & 0 & \lambda & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$\det(M) = 0$ then solve for λ.

The λ's tell you how the matrix is expanding or shrinking vectors.

$\boxed{\text{Example}}$

① Find the eigenvalues of $\underset{A}{\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}}$.

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix} = (1-\lambda)(-3-\lambda) - 2 \cdot 0$$

$$= (1-\lambda)(-3-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 1 \text{ or } -3.}$$

② Same for $\underset{A}{\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}}$,

$$M = A - \lambda I = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ -2 & -4-\lambda \end{pmatrix}$$

$$\det M = (1-\lambda)(-4-\lambda) - 2(-2)$$
$$= (1-\lambda)(-4-\lambda) + 4 = 0$$
$$= -4 - \lambda + 4\lambda + \lambda^2 + 4 = 0$$
$$\Rightarrow \lambda^2 + 3\lambda = 0 \Rightarrow \lambda(\lambda+3) = 0$$
$$\boxed{\lambda = 0 \text{ or } -3}$$

③ Eigenvalues of $A := \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$.

$$A - \lambda I = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 5-\lambda & 0 & 0 \\ 0 & 2-\lambda & 3 \\ 0 & 3 & 2-\lambda \end{pmatrix} = M$$

$$\det M = 0$$

$$= (5-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} - 0\left|\,\right| + 0\left|\,\right|$$

$$= (5-\lambda)\big((2-\lambda)(2-\lambda) - 9\big) = 0$$

$$\Rightarrow (5-\lambda)(4-2\lambda-2\lambda+\lambda^2-9) = 0$$

$$\Rightarrow (5-\lambda)(\lambda^2-4\lambda-5) = 0$$

$$\Rightarrow (5-\lambda)(\lambda-5)(\lambda+1) = 0$$

$$\Rightarrow \lambda = 5, 5, -1$$

double eigenvalue of 5,
single eigenvalue of -1

Symmetric matrix $\begin{pmatrix} 1 & 2 & 7 \\ 2 & -1 & 3 \\ 7 & 3 & 0 \end{pmatrix}$ ← you always have real eigenvalues

Connection with types of critical points of fcns of several variables

<u>Hessian matrix</u> for $F : \mathbb{R}^n \to \mathbb{R}$

$F(x, y, \dots) \in \mathbb{R}$

matrix of 2nd derivatives.

$$H = \begin{pmatrix} F_{xx} & F_{xy} & \cdots \\ F_{yx} & F_{yy} & \cdots \\ F_{zx} & F_{zy} & \cdots \\ \vdots & & \end{pmatrix}$$

F_{yx} means $\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y}\right)$.

<u>Thm</u>: If a function F is C^2 (all 2nd derivatives are continuous), then the mixed partial derivatives are equal.

ie $F_{xy} = F_{yx}$, $F_{zx} = F_{xz}$ etc.

<u>Example</u>: Find the Hessian matrix of

$$F(x,y) = \sin(xy^2) - 2y^2x$$

$$F_x = \cos(xy^2)\cdot y^2 - 2y^2$$

$$F_{xx} = -y^2\sin(xy^2)\cdot y^2 = -y^4\sin(xy^2).$$

$$F_{xy} = -\sin(xy^2)\cdot(2xy)\cdot y^2 + \cos(xy^2)\cdot 2y - 4y$$

$$= -2xy^3\sin(xy^2) + 2y\cos(xy^2) - 4y.$$

$$F_y = \cos(xy^2)(2xy) - 4yx$$

$$F_{yx} = F_{xy}$$

$$F_{yy} = -\sin(xy^2)(2xy)(2xy) + \cos(xy^2)(2x) - 4x.$$

$$H = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix}$$